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Euclidean position in Euclidean 2-orbifolds ☆

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Abstract

Intuitively, a set of sites on a surface is in Euclidean position if points are so close to each other that planar algorithms can be easily adapted in order to solve most of the classical problems in Computational Geometry. In this work we formalize a definition of the term “Euclidean position” for a relevant class of metric spaces, the Euclidean 2-orbifolds, and present methods to compute whether a set of sites has this property. We also show the relation between the convex hull of a point set in Euclidean position on a Euclidean 2-orbifold and the planar convex hull of the inverse image (via the quotient map) of the set.

Keywords: Euclidean position; Metrically convex hull; Orbifold

1. Introduction

There exist many applications of Computational Geometry in which the input and/or output data are given on a surface other than the plane. It is generally assumed in those applications that if a given set is contained on a small portion of the surface, then simple adaptations of planar algorithms (in order to obtain, for instance, the convex hull, the Voronoi diagram or a triangulation with nice properties) can be given. But we are not aware of a general framework for approaching the problem of deciding for which data planar methods are still valid. The only steps in that direction are those given in [6], defining and working with a new concept, the *Euclidean position*, but it is restricted to very specific surfaces such as the cylinder, the torus, the cone and the sphere. It is the aim of this work to generalize that concept to

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a broad class of spaces, which are called *Euclidean* 2-*orbifolds*. The paper is organized as follows. In Section 2 we give definitions, describe the Euclidean 2-orbifolds and establish the notation that will be used along this paper. The definition of Euclidean position together with the theorem that will allow us to work with planar copies of the sets is introduced in Section 3. The relation between the convex hull of a set on a 2-orbifold and the convex hull of one of its planar copies will be set in Section 4. Section 5 presents an algorithm to determine whether a set is in Euclidean position. We conclude in Section 6 with some possible extension of this concept to the remainder surfaces.

2. Preliminaries

As for prerequisites, the reader is expected to be familiar with subjects in [12], but in order to facilitate access to the individual topics, the paper is rendered as self-contained as possible. Thus, in this section we fix the notation and introduce the basic definitions that will be used throughout the paper.

The set of planar motions is a group under the map composition, denoted by $Mo(\mathbb{R}^2)$. The *orbit* of a point $P \in \mathbb{R}^2$ under the action of a discrete group of motions $\Gamma \subseteq Mo(\mathbb{R}^2)$ is the set formed by the images of P via the elements of Γ, $\Gamma P = \{g(P)\colon\ g \in \Gamma\}$. By identifying points at the same orbit, the *quotient space* $S = \mathbb{R}^2/\Gamma$ can be constructed (Fig. 1) and if φ denotes the quotient map,

$$\varphi : \mathbb{R}^2 \to S = \mathbb{R}^2/\Gamma,$$

the orbit ΓP can be also written as $\varphi^{-1}(p) = \{P',\ P'',\ P''', \ldots\}$, for $p \in S = \mathbb{R}^2/\Gamma$.

A convenient way to visualize the orbit space $S = \mathbb{R}^2/\Gamma$ is to focus on a *fundamental domain*, that is a part of the plane which contains a representative of each orbit with at most one representative of each orbit in its interior. If double points (points on the boundary) of a fundamental domain are deleted and its φ-image is considered, we obtain what we call a *fundamental region*. If $P \in \mathbb{R}^2$ is not a fixed point for any motion in Γ (i.e., $\varphi(P)$ is not a *singular point* of S [12]), then the region

$$V_{\Gamma P}(P) = \bigl\{Q \in \mathbb{R}^2\colon\ d(Q, P) \leqslant d\bigl(Q, g(P)\bigr)\ \forall g \in \Gamma\bigr\},$$

where d denotes the euclidean distance in $\mathbb{R}^2$, is a fundamental domain (see [5,9] for a proof) which is called a *Dirichlet domain*. Notice that $V_{\Gamma P}(P)$ is the topological closure of the Voronoi region of P in relation to its orbit and hence, it is convex. The quotient space inherits a metric from the plane:

$$d_S(p, q) = d_S(\Gamma P, \Gamma Q) = \min\bigl\{d(P', Q') \mid P' \in \Gamma P,\ Q' \in \Gamma Q\bigr\}.$$

The right-hand side also equals $\min\{d(P, Q') \mid Q' \in \Gamma Q\}$, because each $P' \in \Gamma P$ has the same set of distances to the members of ΓQ. The latter expression shows that d_S is well-defined because for each

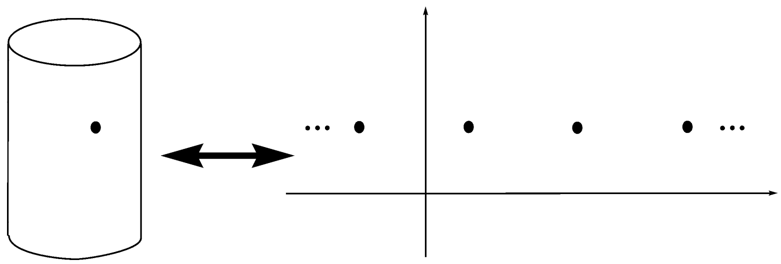

Fig. 1. A point on the cylinder and its orbit on the plane.

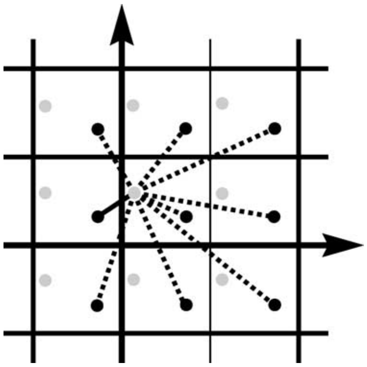

Fig. 2. The segment joining p and q on $S = \mathbb{R}^2/\Gamma$ is the φ-image of the shortest line segment (shown in dark) among those matching one element of ΓP with all the elements in ΓQ.

$P \in \mathbb{R}^2$ there is at least a nearest $Q' \in \Gamma Q$. Hence, it is natural to define *the segment joining* p *and* q on $S = \mathbb{R}^2/\Gamma$ to be the φ-image of $\overline{PQ'}$, which is the shortest line segment among those matching one element of ΓP with all the elements in ΓQ.

As we have pointed out in the introduction, the first reference to the term *Euclidean position* goes back to the work of Grima and Márquez [6], where a point set on the cylinder, the torus, the cone, or the sphere is said to verify the *Local Euclidean Position* property (LEP property for short) if

(1) it is contained between opposite generatrices of the cylinder or the cone;
(2) it is contained in a quadrant (the region between two opposite parallels and two opposite meridians) of the torus;
(3) it is contained in an hemisphere of the sphere.

In that work, it was proven that planar algorithms for computing several geometric structures (such as convex hulls or Voronoi diagrams) are also valid on the respective surfaces if the point set verifies the LEP property. Obviating the case of the sphere, we will take the definition above as the starting point to generalize the LEP property to the 2-orbifolds.

3. Euclidean position

A set $\mathcal{A} \subseteq S = \mathbb{R}^2/\Gamma$ is said to be in *Euclidean position* if there exists a fundamental region containing all segments joining pairs of points in $\mathcal{A}$. In Section 5 we will see that this definition is in agreement with the LEP property when it is restricted to the cylinder, the cone, or the torus. Although the definition of *Euclidean position* refers to sets $\mathcal{A} \subseteq S = \mathbb{R}^2/\Gamma$, it would be useful to obtain a characterization of the property in relation to the inverse image of the set $\varphi^{-1}(\mathcal{A})$. But, note that $\varphi^{-1}(\mathcal{A})$ is constituted by whole orbits. Therefore, with the purpose of choosing a suitable representative of each orbit, we define a *planar copy* of $\mathcal{A}$ to be any of the sets $\widehat{\mathcal{A}} = \varphi^{-1}(\mathcal{A}) \cap D_P$, with D_P being the Dirichlet domain of a point $P \in \varphi^{-1}(\mathcal{A})$. The point of the orbit of $\mathcal{A}$ selected to construct the planar copy is not relevant in order to determine whether the set is in Euclidean position, as is shown in the following theorem.

Theorem 1. *Let $\mathcal{A} \subseteq S = \mathbb{R}^2/\Gamma$ be a point set without singular points. Then the following assertions are equivalent*:

(a) *$\mathcal{A}$ is in Euclidean position,*
(b) *for any $p \in \mathcal{A}$, there does not exist a polygonal chain joining any two points $P, P' \in \varphi^{-1}(p)$ with vertices on $\varphi^{-1}(\mathcal{A})$ and such that the φ-image of each edge is the segment joining the φ-images of its ends,*
(c) *every planar copy $\widehat{\mathcal{A}}$ of $\mathcal{A}$ is contained in the intersection of the open Dirichlet domains of its points.*

Proof. (a) $\Rightarrow$ (b) Suppose, contrary to our claim, that it is possible to find a polygonal chain C joining P and P' under the conditions stated in (b). Since P and P' belong to the same orbit, a fundamental domain containing P and P' on the boundary cannot give rise to a fundamental region containing $p = \Gamma P$. On the other hand, if P is strictly contained inside a fundamental domain, there exists an edge e of C that crosses its boundary, and therefore the corresponding fundamental domain does not contain the segment joining the ends of $\varphi(e)$, which contradicts the fact that $\mathcal{A}$ is in Euclidean position.

(b) $\Rightarrow$ (c) Let $\widehat{\mathcal{A}} = \varphi^{-1}(\mathcal{A}) \cap D_P$ be a planar copy of $\mathcal{A}$, with D_P being the Dirichlet domain of a point $P \in \varphi^{-1}(\mathcal{A})$. First of all, note that there is no point of $\widehat{\mathcal{A}}$ on the boundary of D_P; otherwise, a polygonal chain joining P and another representative P' of its orbit could be found, contrary to (b). This proves that $\widehat{\mathcal{A}} \subset D_P^\circ$, where D_P° denotes the interior of D_P. Now, we fix $Q \in \widehat{\mathcal{A}}$ and prove that $\widehat{\mathcal{A}} \subset D_Q^\circ$. It is obvious that $P \in D_Q$ since $Q \in D_P$. Moreover, if any other point $R \in \widehat{\mathcal{A}}$ $(R \neq P)$ is not in D_Q, there must exist $Q' \in \Gamma Q$ such that $d(R, Q') < d(R, Q)$ and hence, $QPRQ'$ is a polygonal chain that contradicts the hypothesis.

(c) $\Rightarrow$ (a) Let $\widehat{\mathcal{A}}$ be a planar copy of $\mathcal{A}$ and $p \in \mathcal{A}$; by the hypothesis, $\widehat{\mathcal{A}} \subset D_P^\circ$, with $P \in \widehat{\mathcal{A}} \cap \varphi^{-1}(p)$. Now, given $q \in \mathcal{A}$, there exists $Q \in \varphi^{-1}(q) \cap \widehat{\mathcal{A}} \subset D_P^\circ$. Since both P and Q are in D_P°, and the Dirichlet domains are convex sets, the segment $\varphi(P)\varphi(Q) = \overline{pq}$ is strictly contained in $\varphi(D_P^\circ)$, which is a fundamental region. $\square$

The theorem above is restricted to sets without singular points, since Dirichlet domains are defined only for non-fixed points. Nevertheless, since fixed points are either rotation centers or points on the axis of a reflexion, which are always on the boundary of any fundamental domain, their φ-image (which are the singular points) cannot be contained in a fundamental region. Thus, a set containing such points cannot be in Euclidean position as is asserted in the following proposition.

Proposition 1. *A set on a surface $S = \mathbb{R}^2/\Gamma$ containing at least two points, one of them being a singular point of S, cannot be in Euclidean position.*

Once we have defined the term Euclidean position, and the relation between a point set having this property and its planar copies has been verified, it is time to establish the correspondence between the convex hulls of both the set and its copies.

4. Euclidean position and convex hull

The convex hull is one of the most relevant structures in Computational Geometry, and according to our objectives, it is a good test to check the "planar behaviour" of sets in Euclidean position. This motivates the study of the relation between the convex hull of a set in Euclidean position on an Euclidean 2-orbifold and the convex hull of its inverse image on the plane. The planar concept of convexity has been generalized to surfaces in several ways [7,8]. Bringing up the definition given in [6,11], a set $\mathcal{A} \subseteq S = \mathbb{R}^2/\Gamma$ is said to be *metrically convex* if the segment between any two points of $\mathcal{A}$ also lies in $\mathcal{A}$. The *metrically convex hull* $CH_S(\mathcal{A})$ (*convex hull*, henceforth) is defined as the smallest metrically convex set containing $\mathcal{A}$. It can be easily shown that, as in the plane, $CH_S(\mathcal{A})$ can be obtained by intersecting all the convex sets containing $\mathcal{A}$.

The next proposition shows that if a set is in Euclidean position on a surface then it really behaves as a planar set, to the effect that it is isometric to one of its planar copies. We leave to the reader the details of the proof due to its simplicity, but it should be clear that the second assertion is a direct consequence of the first one, and the former can be easily deduced from Theorem 1.

Proposition 2. *Let $\widehat{\mathcal{A}}$ be a planar copy of a set $\mathcal{A} \subseteq S = \mathbb{R}^2/\Gamma$. Then, the following assertions hold*:

(1) *$\mathcal{A}$ is in Euclidean position if and only if φ restricted to $\widehat{\mathcal{A}}$, $\varphi|_{\widehat{\mathcal{A}}}$, is an isometry.*
(2) *If $\mathcal{A}$ is in Euclidean position, then $\mathcal{A}$ is convex if and only if $\widehat{\mathcal{A}}$ is convex.*

Notice that although $\mathcal{A} \subseteq S = \mathbb{R}^2/\Gamma$ is convex, the connected components of $\varphi^{-1}(\mathcal{A})$ need not be this way, as is shown in Fig. 3.

Before dealing with the main theorem in this section, we prove a preliminary result and set the following notation that will be used henceforth. We will use the term *extreme points* of a point set $\mathcal{B}$ (either in $\mathbb{R}^2$ or on $S = \mathbb{R}^2/\Gamma$) to refer to points of the set which are on the boundary of its convex hull, that is, $\partial(CH(\mathcal{B})) \cap \mathcal{B}$, denoted $EXT(\mathcal{B})$.

Proposition 3. *Let $\widehat{\mathcal{A}} = \varphi^{-1}(\mathcal{A}) \cap D_P$ be a planar copy of a point set $\mathcal{A} \subseteq S = \mathbb{R}^2/\Gamma$, being $P \in \varphi^{-1}(\mathcal{A})$. Then, the following assertions are equivalent*:

(a) *$\varphi(EXT(\widehat{\mathcal{A}}))$ is in Euclidean position,*
(b) *$\mathcal{A}$ is in Euclidean position,*
(c) *$CH_S(\mathcal{A})$ is in Euclidean position.*

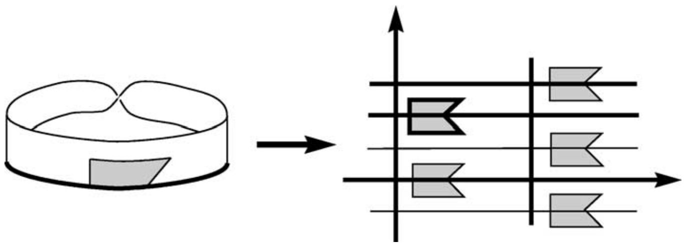

Fig. 3. The connected components in the plane of a convex set in the Möebius strip can be non-convex.

Proof. (a) $\Rightarrow$ (b) Assume that $\varphi(EXT(\widehat{\mathcal{A}}))$ is in Euclidean position; then, there must exist a fundamental domain D such that $EXT(\widehat{\mathcal{A}}) \subset D^\circ$, and by convexity in $\mathbb{R}^2$, $\widehat{\mathcal{A}} \subset D^\circ$. Then, $\mathcal{A} = \varphi(\widehat{\mathcal{A}}) \subset \varphi(D^\circ)$, which is a fundamental region on $S = \mathbb{R}^2/\Gamma$ and this completes the proof.

(b) $\Rightarrow$ (c) First of all we prove that given $R, Q \in CH_{\mathbb{R}^2}(\widehat{\mathcal{A}})$, then R is the nearest representative in ΓR to Q. Otherwise, $Q \notin D_R$ and since the Dirichlet domains are convex sets, there must exists $U \in EXT(\widehat{\mathcal{A}})$ such that $U \notin D_R$ or, equivalently, $R \notin D_U$. Reasoning as before, there must also exist $V \in EXT(\widehat{\mathcal{A}})$ such that $V \notin D_U$. We have then found points $U, V \in \widehat{\mathcal{A}}$ such that $V \notin D_U$, and by Theorem 1, $\mathcal{A}$ is not in Euclidean position, contrary to the hypothesis.

Note that we have actually proved that $\varphi|_{CH_{\mathbb{R}^2}(\widehat{\mathcal{A}})}$ is an isometry and therefore, $\varphi(CH_{\mathbb{R}^2}(\widehat{\mathcal{A}}))$ is in Euclidean position, by Proposition 2. Moreover, since $CH_{\mathbb{R}_2}(\widehat{\mathcal{A}})$ is convex, the same proposition states that so $\varphi(CH_{\mathbb{R}_2}(\widehat{\mathcal{A}}))$ is.

Then, $\varphi(CH_{\mathbb{R}_2}(\widehat{\mathcal{A}}))$ is a set in Euclidean position which contains $CH_S(\mathcal{A})$ (since it is convex and contains $\mathcal{A} = \varphi(\widehat{\mathcal{A}})$), and this proves that $CH_S(\mathcal{A})$ is also in Euclidean position.

(c) $\Rightarrow$ (a) It suffices to show that $\varphi(EXT(\widehat{\mathcal{A}})) \subseteq CH_S(\mathcal{A})$, but this is quite obvious to prove since $EXT(\widehat{\mathcal{A}}) \subseteq \widehat{\mathcal{A}}$ and, therefore, $\varphi(EXT(\widehat{\mathcal{A}})) \subseteq \varphi(\widehat{\mathcal{A}}) = \mathcal{A} \subset CH_S(\mathcal{A})$. $\square$

The next theorem establishes that the convex hull of a set of sites in Euclidean position on an Euclidean 2-orbifold can be obtained as the φ-image of the convex hull of one of its planar copies. Fig. 4 explains the result.

Theorem 2. *Let $\widehat{\mathcal{A}} = \varphi^{-1}(\mathcal{A}) \cap D_P$ be a planar copy of a point set $\mathcal{A} \subseteq S = \mathbb{R}^2/\Gamma$ in Euclidean position, being $P \in \varphi^{-1}(\mathcal{A})$. Then*

$$CH_S(\mathcal{A}) = \varphi\big(CH_{R^2}(\widehat{\mathcal{A}})\big).$$

Proof. Let us begin by noting that proof of (b) $\Rightarrow$ (c) in Proposition 3 shows that $\varphi(\widehat{\mathcal{A}}) = \varphi(CH_{\mathbb{R}^2}(\varphi^{-1}(\mathcal{A}) \cap D_P))$ is in Euclidean position and contains $CH_S(\mathcal{A})$, so the result is stated by showing the other inclusion.

The set $\varphi^{-1}(CH_S(\mathcal{A})) \cap D_P$ is convex by Proposition 2, since it is a planar copy of $CH_S(\mathcal{A})$ (note that $P \in \varphi^{-1}(\mathcal{A}) \subseteq \varphi^{-1}(CH_S(\mathcal{A}))$), which is also a convex set, and it is in Euclidean position by Proposition 3. Moreover, $\varphi^{-1}(CH_S(\mathcal{A})) \cap D_P$ contains $\varphi^{-1}(\mathcal{A}) \cap D_P = \widehat{\mathcal{A}}$, and therefore $CH_{\mathbb{R}^2}(\widehat{\mathcal{A}}) \subseteq$

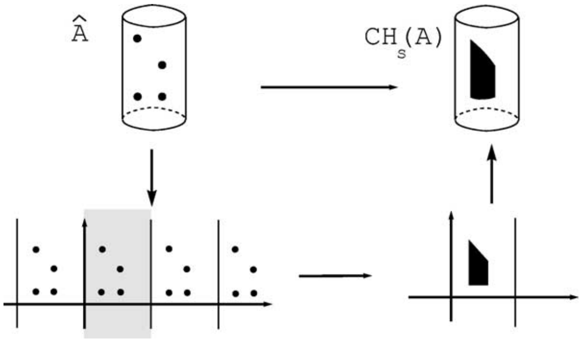

Fig. 4. The convex hull of a set of sites in Euclidean position on S can be obtained as the φ-image of the convex hull of one of its planar copies.

$\varphi^{-1}(CH_S(A)) \cap D_P$; hence, $\varphi(CH_{\mathbb{R}^2}(\widehat{\mathcal{A}})) \subseteq \varphi(\varphi^{-1}(CH_S(A)) \cap D_P) = CH_S(\mathcal{A})$ and the theorem follows. □

5. Determining the Euclidean position

Theorem 2 establishes the relation between the convex hull of a set of sites in Euclidean position $\mathcal{A} \subseteq S = \mathbb{R}^2/\Gamma$ and any of its planar copies $\widehat{\mathcal{A}}$. Roughly speaking, it says that $CH_S(\mathcal{A})$ can be computed as the φ-image of $CH_{\mathbb{R}^2}(\widehat{\mathcal{A}})$, if $\mathcal{A}$ is in Euclidean position.

In this context, to decide whether a point set is in Euclidean position becomes a very important task. We begin this section by proving that the definition of Euclidean position coincides with the LEP property previously introduced in [6] when it is restricted to the cylinder, the cone, or the torus. With this aim, we recall that Theorem 1 reduces this problem to checking if a planar copy of the set lies in the intersection of the open Dirichlet domains of its points.

We first consider the group generated by a single translation, say in the horizontal direction, that gives rise to the cylinder; then, the Dirichlet domain of any point is a parallel-sided strip with fixed width (Fig. 5(a)). Therefore, the only restriction for a set to be in Euclidean position is that its planar copies be contained inside a vertical strip of width half of the modulo of the translation. For any wider set, the Dirichlet domain of the leftmost site does not contain the rightmost's and the φ-image of this strip is just the region between two opposite generatrices on the cylinder. Similar arguments can be applied to the cone, which is generated from a single rotation (Fig. 5(b)).

This study can be extended to the flat torus generated, as usual, by two orthogonal translations with the same modulo. In this case, the Dirichlet domains are squares, and mimicking the reasoning followed in the cylinder, both in the horizontal and in the vertical directions, it is easily seen that the condition for a set to be in Euclidean position is that its planar copies must be contained inside a square of size half of the modulo of the translations, which corresponds (via φ) with the region included between two opposite parallels and two opposite meridians on the surface.

An optimal $\theta(N)$ algorithm to determine whether a set of N points is in Euclidean position on the cylinder, the cone, or the torus is developed in [6]. It checks if the orthogonal projection of the set on a circle is contained in a covering arc of length lesser than π.

Comments above are summarized in the next theorem.

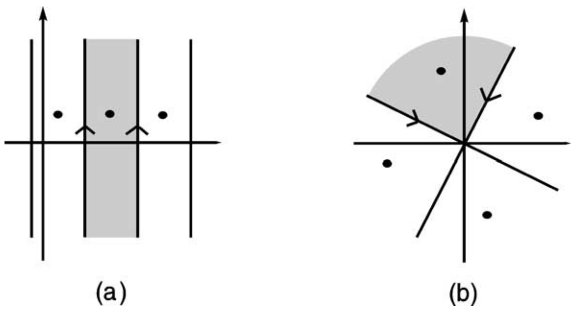

Fig. 5. Dirichlet domain of a point for a group generated by a single (a) translation or (b) rotation.

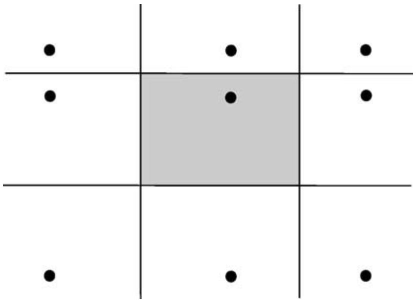

Fig. 6. The orbit of a point on a surface generated by two reflections with orthogonal axis.

Theorem 3. *A set $\mathcal{A}$ of N points on the cylinder, the cone, or the torus is in Euclidean position if and only if it verifies the LEP property. Moreover, it takes $\theta(N)$ time to decide if $\mathcal{A}$ is in Euclidean position on any of these surfaces.*

It is worthwhile to point out here that the case of the torus generated from two non-orthogonal translations will be included in the general result given in Theorem 4, since the different configuration of the generators changes some of the properties involving the metric of the surface [2].

We have already seen that the equivalence with the LEP property provides optimal algorithms to check the Euclidean position of point sets on surfaces generated from a single translation (the cylinder), a single rotation (the cone), or two orthogonal translations (the torus). Now, we turn our attention to point sets on surfaces generated only by reflections, as depicted in Fig. 6.

The bisector of a point and its image by a reflection is the axis of the reflection. Moreover, seen in Fig. 6, the Dirichlet domain has the same shape for any non-fixed point (that is, a point which is on neither of the axes), and it is easy to check that for any $p, q \in \mathcal{A}$ and $P \in \varphi^{-1}(p)$, the representative of ΓQ which is the nearest to P is always at the same half-planes (defined by the axis) as P. As a consequence, the following proposition holds.

Proposition 4. *A set $\mathcal{A}$ of N points on a surface $S = \mathbb{R}^2/\Gamma$, where Γ is generated by one reflection, is in Euclidean position if and only if the axis of the reflection does not split any planar copy $\widehat{\mathcal{A}}$ of $\mathcal{A}$. Moreover, it takes $\theta(N)$ time to decide if $\mathcal{A}$ is in Euclidean position on this surface.*

To make our work complete, it remains to consider surfaces which are generated from glide reflections. The simplest case is the twisted cylinder, which is generated from the composition of one reflection and one translation such that it reflects in the x-axis and translates the x-axis by distance 1.

With the aim of finding efficient algorithms to test if a point set on the twisted cylinder is in Euclidean position, our first attempt is to combine characterizations given for surfaces generated separately from both a translation and a reflection. But, arguments followed for surfaces generated by a reflection fail here, since examples can be found of point sets on the twisted cylinder which are in Euclidean position and such that any of their planar copies are traversed by the axis of the reflection, as we will see below.

The simple behaviour of surfaces generated by single translations is not maintained when the generator is a glide reflection. Notice that the LEP property involves the existence of “maximal regions” for the Euclidean position, in the sense that they are not contained in any larger one having that property and for the moment, we have not been able to characterize such king of regions on the twisted cylinder. Moreover, we have found three different families, two of them consisting of an infinite number of such "maximal regions", and they are not even an exhaustive list since point sets can be constructed in Euclidean position which are not strictly contained inside any of them, as we see immediately after.

The shape of the Dirichlet domains in the case of the twisted cylinder strongly depend on the location of the point considered. Set an orthogonal coordinate system whose x-axis is the axis of the glide reflection and it is translated by distance 1; then, the orbit of a point (x, y) is the set $\{(x+n, (-1)^{|n|}y);\ n \in \mathbb{Z}\}$, and the fundamental regions are vertical parallel-sided strips of width 1 in which double points on opposite sides of the boundary are identified by a twist. Given a point $P(a, b)$ $(b \neq 0)$, the bisector of the segment from P to its image by a glide reflection $g(P) = (a+1, -b)$ is the straight line

$$r_r(P):\ \ y = \frac{1}{2b}\left(x - \left(a + \frac{1}{2}\right)\right).$$

Similarly,

$$r_l(P):\ \ y = -\frac{1}{2b}\left(x - \left(a - \frac{1}{2}\right)\right)$$

is obtained as the bisector of P and $g^{-1}(P) = (a-1, -b)$. Both $r_r(P)$ and $r_l(P)$ intersect the OX axis in $(a + \frac{1}{2}, 0)$ and $(a - \frac{1}{2}, 0)$, respectively, and their slopes only depend on the y-coordinate of P.

Lines $r_r(P)$, $r_l(P)$, $x = a + 1$ (that bisects the segment from $P(a, b)$ to $g^2(P) = (a+2, b)$) and $x = a - 1$ (that bisects the segment from $P(a, b)$ to $g^{-2}(P) = (a-2, b)$) constitute the boundary of the Dirichlet domain of P. Some Dirichlet domains are depicted in Fig. 7.

Now, let D be the fundamental domain having as sides the lines $x = 0$ and $x = 1$, and consider the portion of the curves u_1: $x = \frac{1}{4} - y^2$ and u_2: $x = \frac{3}{4} + y^2$ which lie inside D (see Fig. 8). For any two points $P \in u_1 \cap D$ and $Q \in u_2 \cap D$ with the same y-coordinate, let s_1 (respectively s_2) be the segment of the tangent line to u_1 (respectively to u_2) on P (respectively Q) with end points P (respectively Q) and its point of intersection with the x-axis. For each pair (P, Q), the φ-image on the twisted cylinder of the open region bounded by the vertical half-lines rooted at points P and Q, the segments s_1 and s_2 and the x-axis (the shaded region in Fig. 8) is a maximal region for the Euclidean position. We denote the family constituted by these sets as $\mathcal{F}_1$.

A second family $\mathcal{F}_2$ of maximal regions can be built as the φ-images of the open regions bounded by the intersection of the angular sectors delimited by the tangent lines to u_1 and u_2 on P and Q and lines joining these points with $(0, 0)$ and $(0, 1)$, respectively (Fig. 9(a)).

Finally, $\mathcal{F}_3$ is constituted by the $\varphi-$images of the translations in the horizontal direction of the open set delimited by the curves u_1: $x = \frac{1}{4} - y^2$ and u_3: $x = -\frac{1}{4} + y^2$ (Fig. 9(b)).

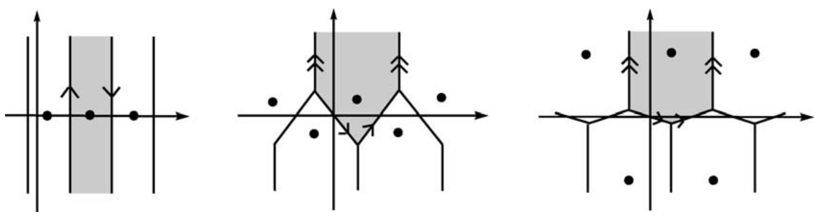

Fig. 7. The shape of the Dirichlet domain in groups generated by a glide reflection strongly depends on the height of the point.

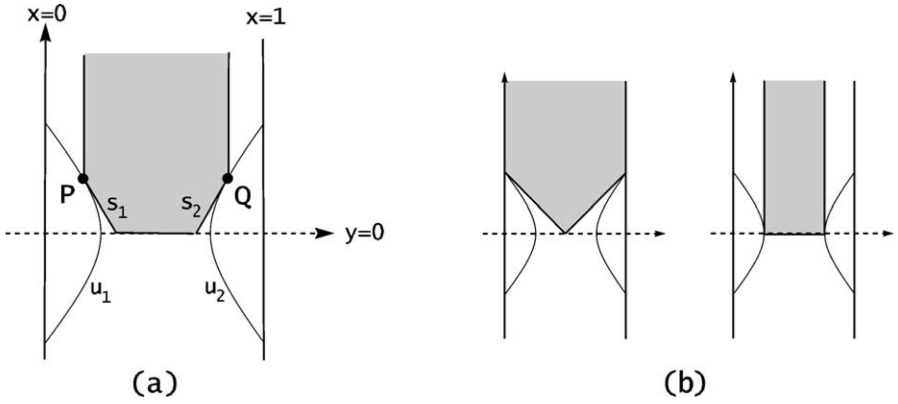

Fig. 8. (a) Constructing sets of the family $\mathcal{F}_1$ and (b) the two extreme cases.

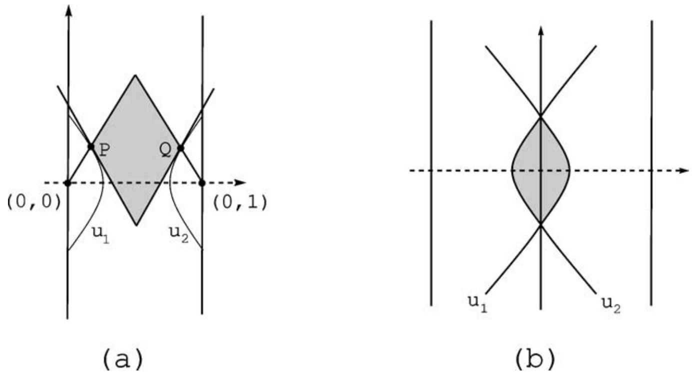

Fig. 9. Planar copy of a set of (a) $\mathcal{F}_2$ and (b) $\mathcal{F}_3$.

Proposition 5. *The sets of $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ are maximal for the Euclidean position on the twisted cylinder. Moreover, they do not constitute an exhaustive list. That is, sets can be found in Euclidean position which are contained in neither of these families.*

Proof of Proposition 5 is laborious (on account of the bothersome calculations) rather than difficult. In order to shorten the paper, we have not included here the detailed verifications but we refer the reader to an extended version of the paper which can be found in [3]. Note that members of families $\mathcal{F}_2$ and $\mathcal{F}_3$ are examples of sets in Euclidean position which are traversed by the axis of the reflection.

Since we have not been able to characterize the maximal regions for the Euclidean position on the twisted cylinder (and, as a consequence, for any surface generated by a group containing a reflection glide) we have to develop new methods to provide an algorithm which check whether a point set on this surface is in Euclidean position. Our algorithm will take $O(N \log N)$ time, more expensive than the linear time needed for groups generated by other motions.

Let $\mathcal{A} = \{p_1, \ldots, p_N\}$ be a set on the twisted cylinder and $\widehat{\mathcal{A}} = \{P_1, \ldots, P_N\}$ one of its planar copies. We construct the convex hull of $\widehat{\mathcal{A}}$, $CH_{\mathbb{R}^2}(\widehat{\mathcal{A}})$, and we denote $EXT(\widehat{\mathcal{A}}) = \{P_{i_1}, \ldots, P_{i_H}\}$ the set of extreme points of $\widehat{\mathcal{A}}$ sorted clockwise. By Proposition 3, in order to know if $\mathcal{A}$ is in Euclidean position, it suffices to check if $EXT(\widehat{\mathcal{A}})$ is inside the intersection of the open Dirichlet domains.

Notice that since the Dirichlet domain of any point is always contained inside a vertical band two units wide centered at the point, the vertical width of the convex hull has to be smaller than or equal to one if the set is in Euclidean position.

Now, we will split $EXT(\widehat{\mathcal{A}})$ into several polygonal chains. First of all, the four vertices having one of the smallest or the largest coordinates (the top, the bottom, the right-most and the left-most vertices) split the convex hull into at most four monotone polygonal chains (we denote by *tr*, *tl*, *bl* and *br* those polygonal chains; see Fig. 10). Additionally, we split into two polygonal chains any of those four chains that is intersected by the axis of the glide reflection, that for the sake of simplicity we will suppose is the OX axis. For instance, if the chain *tr* contains points at both sides of the axis, the two new polygonal chains obtained will be denoted by *tm* and *mr*. In this way, we can obtain up to six polygonal chains, each one of them being a monotone chain with all its points at the same side of the axis of the glide reflection.

The next step is to associate a partition of the OX axis with each one of the previous polygonal chains. This partition will be determined by the intersection points of the prolongation of each segment in the polygonal chain with the OX axis. If the polygonal chain intersects the OX axis, that intersection will determine an unbounded interval that will be denoted by $\overline{se}$ (Fig. 11).

Now, we will describe our procedure regarding the chain tl. Let P be a vertex of $EXT(\widehat{\mathcal{A}})$ and $R_l(P)$ the intersection between the OX axis and $r_l(P)$. If $R_l(P)$ belongs to $\overline{se}$, then the Dirichlet domain of P does not contain $EXT(\widehat{\mathcal{A}})$, and $\mathcal{A}$ is not in Euclidean position. First of all, we find the interval of the partition associated with the polygonal where $R_l(P)$ is. Secondly, we consider the straight line joining $R_l(P)$ and the vertex of greatest absolute value ordinate among the three vertices that define the interval

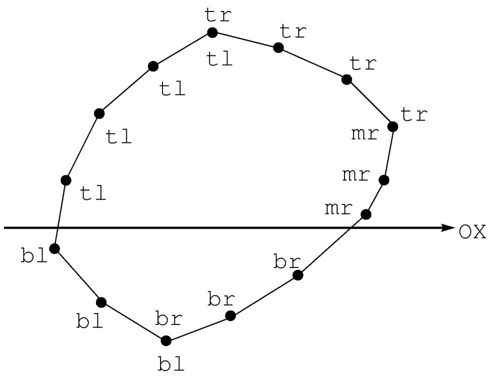

Fig. 10. A convex hull divided in five polygonal chains.

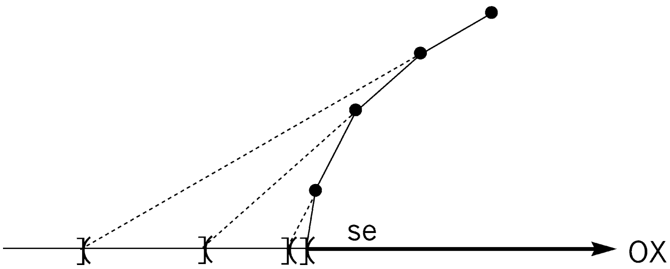

Fig. 11. Partition of the OX axis corresponding to the polygonal *tl*.

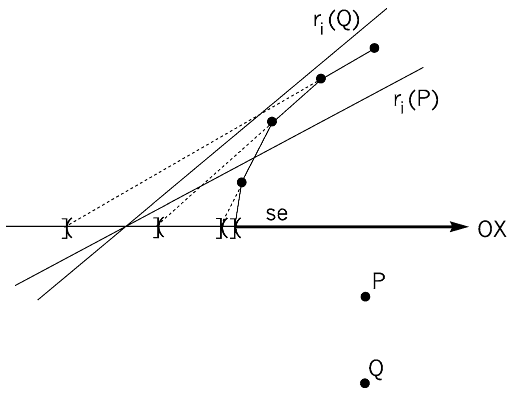

Fig. 12. Comparing the slopes it can be deduced that $r_l(P)$ divides $EXT(\widehat{\mathcal{A}})$ and that it cannot be contained in the Dirichlet domain of P. The opposite situation is given for Q (at least, for this polygonal).

where $R_l(P)$ is. If the slope of this line is greater than the slope of $r_l(P)$, then $r_l(P)$ intersects $EXT(\widehat{\mathcal{A}})$ (Fig. 12) and the set is not in Euclidean position.

Following a similar reasoning for the other polygonal chains (consider $r_r(P)$ instead of $r_l(P)$, when necessary) it can be determined if $EXT(\widehat{\mathcal{A}})$ is inside the Dirichlet domains of its vertices.

The procedure followed above leads to the next algorithm, that decides whether a set of N sites on the twisted cylinder is in Euclidean position in $\mathrm{O}(N \log N)$ time.

Algorithm *EP-TWISTCYL*.
Input: $\mathcal{A} = \{p_1, \ldots, p_N\}$ a set of sites on the twisted cylinder.

(1) Construct a planar copy $\widehat{\mathcal{A}} = \{P_1, \ldots, P_N\}$, of $\mathcal{A}$.
(2) Construct the extreme points of $\widehat{\mathcal{A}}$: $EXT(\widehat{\mathcal{A}}) = \{P_{i_1}, \ldots, P_{i_H}\}$.
(3) Check the width of $EXT(\widehat{\mathcal{A}})$. Is it smaller or equal than one?
  YES $\rightarrow$ Report: $\mathcal{A}$ is not in Euclidean position.
  NO $\rightarrow$ Go to Step 4.
(4) Construct the polygonals *tr*, *tl*, *br*, *bl*, *mr* and *ml*, their induced partitions over the OX axis and the interval $\overline{se}$.
(5) From $j = 1$ to H find $R_l(P_{i_j})$ and $R_r(P_{i_j})$.
  (a) Do they belong to $\overline{se}$?
    YES $\rightarrow$ Report: $\mathcal{A}$ is not in Euclidean position.
    NO $\rightarrow$ Go to Step 5b.
  (b) For each polygonal chain, find the interval of its associated partition in which $R_l(P_{i_j})$ (respectively $R_r(P_{i_j})$) is and compare the slope of $r_l(P_{i_j})$ (respectively $r_r(P_{i_j})$) and that of the straight line joining the point with the corresponding vertex of the convex hull.
  Does the line intersect the hull?
    YES $\rightarrow$ Report: $\mathcal{A}$ is not in Euclidean position.
    NO $\rightarrow$ [j $\rightarrow$ j+1].
(6) Return $\mathcal{A}$ is in Euclidean position.

Thus, we have the following result:

Proposition 6. *It is possible to decide whether a set of N sites on the twisted cylinder is in Euclidean position in $\mathrm{O}(N \log N)$ time.*

Proof. Steps 1 and 4 of the algorithm take linear time. $\mathrm{O}(N \log N)$ time is required by Step 5. Each iteration in Step 2 needs logarithmic time, and N of them are needed in the worst case. So, the whole algorithm takes $\mathrm{O}(N \log N)$ time. □

Note that depending on the position of the vertex it is possible to exclude some of the tests in Step 2. Moreover, if the planar copy is at one side of the axis we only need to consider the polygonal chains *tl* and *tr* if the points have positive ordinate, or *bl* and *br* otherwise.

Once we have studied the simplest cases of discrete groups generated by a single motion, it is time to advance to the general case. In this task we will consider the following question: given a point P, which are the elements of its orbit that can "metrically affect" P (in the sense that they can be Voronoi neighbors of P)? The answer to this question can be found in [10] where it is established that points in a certain fundamental domain are metrically affected only by the elements of the orbits lying in the proper domain or in the finite union of some of its copies.

Lemma 1 [10]. *Given a discrete group of motions Γ and a Dirichlet domain D, there exists a finite subset $\Gamma^* = \{g_1, g_2, \ldots, g_m\}$ of Γ such that for every point $P \in D$ and for every point $Q \in \mathbb{R}^2 - \bigcup_{j=1}^{m} g_j(D)$, there exists another point $Q^* \in \bigcup_{j=1}^{m} g_j(D)$ such that Q^* is in the same orbit that Q and*

$$d(P, Q^*) < d(P, Q).$$

As a consequence it happens that

$$V_{\Gamma P}(P_i) \subset \bigcup_{j=1}^{m} g_j(D).$$

The authors in [10] also prove that m is bounded, and a case analysis yields that $m = 37$ is an upper bound for all possible realizations and all groups.

At this point, we can prove the main result of this section:

Theorem 4. *Given a set $\mathcal{A}$ of N sites on a Euclidean 2-orbifold $S = \mathbb{R}^2/\Gamma$, it is possible to determine if P is in Euclidean position in*

(1) $\Omega(N)$ *time if Γ does not contain a glide reflection*;
(2) $\mathrm{O}(N \log N)$*, otherwise.*

Proof. By Lemma 1, points of a planar copy $\widehat{\mathcal{A}}$ of a set $\mathcal{A}$ in Euclidean position are only affected by the points of $\bigcup_{j=1}^{m} g_j(\widehat{\mathcal{A}})$, with $\{g_1, \ldots, g_m\}$ being a finite subset of the generating group Γ.

We have already described a method to determine if $\widehat{\mathcal{A}}$ is inside the Dirichlet domain of any of its points when any motion is considered. So it only remains to apply this procedures to every motion g_i, $i = 1, \ldots, m$. If the answers to the m tests are affirmative (and recall that $m \leqslant 37$), then it is possible to ensure that $\widehat{\mathcal{A}}$ is not metrically affected by other points of $\varphi^{-1}(\mathcal{A})$ but only by its owns points and, by Theorem 1, we conclude that $\mathcal{A}$ is in Euclidean position.

If g_i is a reflection, a translation or a rotation, this procedure takes $\Omega(N)$ time, and $O(N \log N)$ time is required when it is a glide reflection. □

6. Conclusions and open problems

In this paper, we have generalized to 2-orbifolds the definition of *Euclidean position* previously introduced for a few surfaces by Grima and Márquez in [6]. We have characterized this property and provided algorithms to check whether a point set is in Euclidean position. A natural step now is to try an extension of this notion of planar behavior to other surfaces.

A first approximation is given in [6], where a more general definition is presented in the Algebraic Topology field. More specifically, given a point set $\mathcal{A}$ on a surface, a sequence of sets $G_k(\mathcal{A})$, $k \geqslant 1$ is recursively defined, as the set of segments joining points of $G_{k-1}(\mathcal{A})$, with $G_0(\mathcal{A})$ being the set of segments joining pairs of points in $\mathcal{A}$. This sequence obviously converges to the metrically convex hull $CH(\mathcal{A})$. In [6], the authors propose to set $\mathcal{A}$ to be in Euclidean position if $CH(\mathcal{A})$ is simply connected, that is, if $CH(\mathcal{A})$ does not contains "holes", the definition of which matches with the intuitive idea of planar behavior on a surface (see Fig. 13).

Another possible generalization is by starting from the concept of *cut point* brought from the Differential Geometry. A *cut point* of a point p is the point q such that if we prolongate the shortest geodesic joining p and q, the geodesic so obtained is not longer a minimizing geodesic [1,4]. The set of all the cut points of a given point is called a *cut loci*. Thus, the notion of Euclidean position could be extended as follows: If $\mathcal{A}$ is a set of sites on a surface, we set $\mathcal{A}$ to be in Euclidean position if $G_0(\mathcal{A})$ does not intersect the cut loci of any of its points. If we consider, for instance, the cylinder, it is easy to see that the cut loci of a point is its opposite generatrix [13], hence this new definition agrees with the LEP property introduced in [6]; and it can be checked that the same happens for the cone and the torus.

It remains to check if these generalizations are consistent with that given in this paper for Euclidean 2-orbifolds, and find methods for checking whether a set is in Euclidean position under these new definitions.

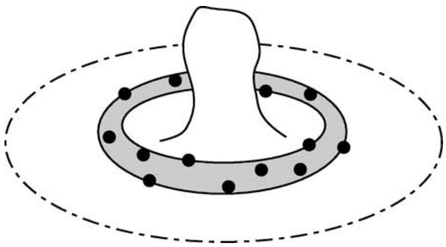

Fig. 13. $CH(\mathcal{A})$ (shaded) has a "hole" due to the high values of the curvature in a certain region of the surface, so $\mathcal{A}$ is not in Euclidean position.

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